MATH 10550
SOLUTIONS TO PRACTICE FINAL EXAM

1. Compute $\lim_{x\to 2^-} \dfrac{x^2-4}{x^2-5x+6}$.

Solution

When $x \neq 2$,

$$\frac{x^2-4}{x^2-5x+6} = \frac{(x-2)(x+2)}{(x-2)(x-3)} = \frac{x+2}{x-3}.$$

So

$$\lim_{x\to 2^-} \frac{x^2-4}{x^2-5x+6} = \frac{2+2}{2-3} = -4.$$

2. Compute $\lim_{x\to 0^+} \dfrac{x^2-9}{\sin x}$.

Solution

If x is close to 0 but larger than 0, then the denominator $\sin x$ is a small positive number and x^2-9 is close to -9. So the quotient $\dfrac{x^2-9}{\sin x}$ is a *large negative* number. So

$$\lim_{x\to 0^+} \frac{x^2-9}{\sin x} = -\infty.$$

3. Evaluate $\lim_{x\to\infty}(\sqrt{x^2-x}-\sqrt{x^2+5x})$.

Solution

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{x^2-x}-\sqrt{x^2+5x}) &= \lim_{x\to\infty}(\sqrt{x^2-x}-\sqrt{x^2+5x})\frac{\sqrt{x^2-x}+\sqrt{x^2+5x}}{\sqrt{x^2-x}+\sqrt{x^2+5x}} \\
&= \lim_{x\to\infty}\frac{(x^2-x)-(x^2+5x)}{\sqrt{x^2-x}+\sqrt{x^2+5x}} \\
&= \lim_{x\to\infty}\frac{-6x}{\sqrt{x^2(1-\frac{1}{x})}+\sqrt{x^2(1+\frac{5}{x})}} \\
&= \lim_{x\to\infty}\frac{-6x}{x\sqrt{1-\frac{1}{x}}+x\sqrt{1+\frac{5}{x}}} \\
&= \lim_{x\to\infty}\frac{-6}{\sqrt{1-\frac{1}{x}}+\sqrt{1+\frac{5}{x}}} \\
&= \frac{-6}{\sqrt{1-0}+\sqrt{1+0}} = -3.
\end{aligned}$$

4.

Let $f(x) = \begin{cases} ax+1 & x<0, \\ x^2+1 & x\geq 0. \end{cases}$

For what constant a is f differentiable everywhere?

Solution

f is clearly differentiable for $x<0$ and for $x>0$. For $x<0$, $f'(x)=a$, so $\lim_{x\to 0^-} f'(x)=a$. For $x>0$, $f'(x)=2x$, so $\lim_{x\to 0^+} f'(x)=0$. For f to be differentiable at 0, we need $a=\lim_{x\to 0^-} f'(x)=\lim_{x\to 0^+} f'(x)=0$.

5. Compute $\lim_{x\to 0} \frac{\tan 2x}{\sin 3x}$.

Solution

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan 2x}{\sin 3x} &= \lim_{x\to 0} \frac{\sin 2x}{\cos 2x \cdot \sin 3x} \\
&= \lim_{x\to 0} \left(\frac{\sin 2x}{x} \cdot \frac{1}{\cos 2x} \cdot \frac{x}{\sin 3x} \right) \\
&= \lim_{x\to 0} \frac{2\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{\cos 2x} \cdot \lim_{x\to 0} \frac{3x}{3\sin 3x} \\
&= 2\lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{1}{\cos 2x} \cdot \frac{1}{3} \lim_{x\to 0} \frac{3x}{\sin 3x} \\
&= 2(1) \cdot 1 \cdot \frac{1}{3}(1) = \frac{2}{3}.
\end{aligned}$$

6. Compute $\lim_{x\to -\infty} \frac{\sqrt{4x^2+x+1}}{3x-1}$.

Solution.

Note $x = -\sqrt{x^2}$ for $x < 0$. Multiply the top and bottom of $\frac{\sqrt{4x^2+x+1}}{3x-1}$ by $\frac{1}{x}$ to get

$$\frac{\sqrt{4x^2+x+1}}{3x-1} = \frac{\frac{1}{x} \cdot \sqrt{4x^2+x+1}}{\frac{1}{x} \cdot (3x-1)} = -\frac{\sqrt{4+(1/x)+(1/x^2)}}{3-(1/x)}.$$

Hence the limit as $x \to -\infty$ is $-\frac{\sqrt{4}}{3} = -\frac{2}{3}$.

7. Compute the tangent line to the ellipse given by the equation $x^2 + 4y^2 = 5$ at the point $(1, -1)$

Solution. If we take the derivative with respect to both sides of the equation we see $2x + 8y\frac{dy}{dx} = 0$, or $\frac{dy}{dx} = -\frac{x}{4y}$. So the slope at $(1, -1)$ is $\frac{1}{4}$. Thus the tangent line at $(1, -1)$ is

$$y + 1 = \frac{1}{4}(x-1),$$

or

$$y = \frac{1}{4}x - \frac{5}{4}.$$

8.Let $F(x) = f(g(x))$. Compute $F'(2)$ using the following information:

$f(-1) = -3$, $f(2) = 12$, $g(-1) = -7$, $g(2) = -1$,

$f'(-1) = 2$, $f'(2) = 8$, $g'(-1) = -1$, $g'(2) = 5$.

Solution. Using the chain rule, $F'(x) = f'(g(x))g'(x)$, so $F'(2) = f'(g(2))g'(2) = f'(-1)\cdot 5 = 2\cdot 5 = 10$

9. For $y = (\sin 4x)^8$, compute y'.

Solution. Using the chain rule a total of three times, we get

$$\begin{aligned} y' =& 8\cdot(\sin 4x)^7\cdot\frac{d}{dx}(\sin(4x)) = 8\cdot(\sin 4x)^7\cdot\cos 4x\cdot\frac{d}{dx}4x \\ =& 32(\sin 4x)^7\cos 4x. \end{aligned}$$

10. How many inflection points does the curve $y = \frac{x^5}{5} + \frac{x^4}{4}$ have?

Solution. First we note $y' = x^4 + x^3$ and $y'' = 4x^3 + 3x^2 = x^2(4x+3)$. Hence $y'' = 0$ at $x = 0$ and $x = -3/4$. However, $y'' < 0$ in $(-\infty, -3/4)$ and $y'' > 0$ in $(-3/4, 0)$ and $(0, \infty)$. Hence only $-3/4$ is the inflection point.

11. Compute the derivative y' for the curve $\sqrt{x^2+y^2} = 2 + y$ at the point $x = 4$, $y = 3$.

Solution. Taking the derivative of both sides of the equation and using the chain rule gives $1/2(x^2+y^2)^{-1/2}(2x+2yy') = y'$. So evaluating at the point $x = 4, y = 3$, we get $1/2(4^2+3^2)^{-1/2}(2\cdot 4 + 2\cdot 3y') = y'$ gives $1/10(8+6y') = y'$ which we can solve for to get $y' = 2$.

12. A kite 100 ft above the ground is flying horizontally (away from its holder) with a speed of 16ft/sec. At what rate is the angle between the string and the horizontal direction changing, when 200 ft of the string have been let out?

Solution. The kite is at a constant height of 100ft with a length of x ft away. So using some trigonometry, we see that if θ is the angle between the string and the horizontal direction, $\tan\theta = 100/x$. Taking the derivative with respect to t gives $\sec^2\theta\frac{d\theta}{dt} = -100x^{-2}\frac{dx}{dt} = -1600x^{-2}$ since the kite is flying away at 16ft/sec. When 200ft have been let out, $\sin\theta = 1/2$ and $\theta = \pi/6$. At this value of θ we have $4/3\frac{d\theta}{dt} = -\frac{1600}{200^2-100^2}$ or

$$\frac{d\theta}{dt} = -\frac{1200}{30000} = -\frac{1}{25}\frac{radians}{second}.$$

13. Find the linearization of $f(x) = \sqrt{10-x^2}$ at $a = -1$.

Solution. The linearization of $f(x)$ at $a = -1$ is $L(x) = f(a) + f'(a)(x-a)$. At $a = -1$, $f(a) = 3$ and $f''(x) = 1/2(10-x^2)^{-1/2}\cdot(-2x)$ so $f'(a) = 1/3$ and substituting back in gives $f(x) = 3 + 1/3(x+1)$.

14. Find all local maxima and minima of the function $f(x) = 2|x| - x^2 - 1$.
Solution. Taking the derivative of $2x - x^2 - 1$ for $x > 0$ gives $f' = 2 - 2x$ which means a critical point is at $x = 1$. Taking the derivative again and using the second derivative test gives $x = 1$ is a local maxima.

Taking the derivative of $-2x - x^2 - 1$ for $x < 0$ gives $f' = -2 - 2x$ which means a critical point is at $x = -1$. Taking the derivative again and using the second derivative test gives $x = -1$ is a local maxima.

When $x < 0$ the function is increasing and when $x > 0$ the function increases for small values away from 0 so it is easy to see that $x = 0$ is a local minima.

15. Find all asymptotes of the curve $y = \frac{2x^2+x+1}{x-1}$.
Solution. The curve $y = \frac{2x^2+x+1}{x-1}$ is undefined at $x = 1$ and so has a vertical asymptote there. Now,

$$\lim_{x \to \pm\infty} \frac{2x^2 + x + 1}{x - 1} = \pm\infty,$$

so there is no horizontal asymptote. Let's do long division to see if there is a slant asymptope. One gets

$$\frac{2x^2 + x + 1}{x - 1} = 2x + 3 + \frac{4}{x - 1}.$$

Hence $y = 2x+3$ is a slant asymptope because as $x \to \pm\infty$, $2x+3+\frac{4}{x-1}$ goes to $2x + 3$.

16. Find **all** the points on the hyperbola $y^2 - x^2 = 4$ that are closest to the point $(2, 0)$.
Solution. The distance formula between a point (x, y) and $(2, 0)$ is given by

$$d(x, y) = \sqrt{y^2 + (x - 2)^2}.$$

We want to consider only points (x, y) such that $y^2 - x^2 = 4$. Solving for y^2, $y^2 = x^2 + 4$. Substituting into the distance formula, we have

$$d(x) = \sqrt{x^2 + 4 + (x - 2)^2} = \sqrt{2x^2 - 4x + 8}.$$

To minimize distance, we use the first derivative to find critical points.

$$d'(x) = \frac{4x - 4}{2\sqrt{2x^2 - 4x + 8}}.$$

If $d'(x) = 0$ then $4x - 4 = 0$ and so $x = 1$. Note $d'(x) < 0$ for $x < 1$ and $d'(x) > 0$ for $x > 1$. Hence $d(x)$ decreases for $x < 1$ and increases for $x > 1$, and therefore $d(x)$ realizes its global minimum at $x = 1$. Because a hyperbola is not an actual function, there can be more than one y associated with a particular x. Earlier we found $y^2 = x^2 + 4$. Then $y = \sqrt{x^2 + 4}$. When $x = 1$, $y = \pm\sqrt{5}$. So the two distance minimizing points on the hyperbola are $(1, \pm\sqrt{5})$.

17. A page of a book is to have a total area of 150 square inches, with 1 inch margins at the top and sides, and a 2 inch margin at the bottom. Find the dimensions in inches of the page which will have the largest print area.

Solution. You should draw a picture for this problem. If l is the total length of the page, w is the total width, and A is the print area, then

$$lw = 150, A = (l - 2)(w - 3).$$

We want to maximize A so we want to substitute in for one of the variables so that we can take the derivative. Note $l = 150/w$. Therefore,

$$A(w) = (\frac{150}{w} - 2)(w - 3) = 150 - \frac{450}{w} - 2w + 6,$$

and

$$A'(w) = \frac{450}{w^2} - 2.$$

If $A'(w) = 0$ then $w^2 = 225$ so $w = 15$. The first derivative test easily shows this gives a maximum area. Since $l = 150/w$, $l = 10$.

18. Newton's method is to be used to find a root of the equation

$$x^3 - x - 1 = 0.$$

If $x_1 = 1$, find x_2.

Solution. If we let $f(x) = x^3 - x - 1$ then $f'(x) = 3x^2 - 1$. Hence

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-1}{2} = 1.5.$$

19. Express the limit below as a definite integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{4n} \sec^2\left(\frac{i\pi}{4n}\right)$$

Solution. It is probably easiest to guess first that the sum is a straightforward Riemann Sum and only look for tricks if necessary. Normally in a Riemann sum to n, n is the number of divisions of the interval. We should also expect the $\frac{\pi}{4n}$ to be the length of the subintervals. Hence the total interval should be of length $\frac{\pi}{4}$. Then it is clear that $\frac{i\pi}{4n}$ is simply the right-hand endpoint of the i-th interval. Hence our limit is simply $\int_0^{\pi/4} \sec^2(x)dx$.

20. If $f(x) = \displaystyle\int_0^{5x} \cos(u^2)du$, find $f'(x)$

Solution. We want to apply the Fundamental Theorem of Calculus, but cannot immediately. So we define

$$g(x) = \int_0^x \cos(u^2)du$$

to get $f(x) = g(5x)$. By the chain rule, $f'(x) = 5g'(x)$. Now we can apply the Fundamental Theorem to g:

$$f'(x) = 5g'(x) = 5\cos((5x)^2) = 5\cos(25x^2).$$

21. Evaluate the integral $\displaystyle\int_0^{\sqrt{\pi}} x\sin(x^2)dx$.

Solution. Let $u = x^2$. Then $du = 2xdx$ and

$$\int_0^{\sqrt{\pi}} x\sin(x^2)dx = \frac{1}{2}\int_0^{\pi} \sin(u)du = -\frac{1}{2}\cos(u)|_0^{\pi} = 1/2 + 1/2 = 1.$$

22. Which of the following integrals give the area of the region below the curve $y = 2x$ and above the curve $y = x^2 - 4x$?

Solution. You should draw a picture. We find the intersection points of the two curves by solving

$$2x = x^2 - 4x \Leftrightarrow x^2 - 6x = 0 \Leftrightarrow x(x-6) = 0 \Leftrightarrow x = 0, 6.$$

For small x, $x^2 - 4x < 0$. In $[0, 6]$, the curve $y = x^2 - 4x$ is below $y = 2x$. This can be seen by taking, say, $x = 1$. Therefore the area between the curves is given by

$$\int_0^6 \Big(2x - (x^2 - 4x)\Big)dx.$$

23. Consider the area in the xy plane bounded by the curves $y = 0$ and $y = x - x^2$. If we rotate this area about $x = 7$, what integral gives the volume?

Solution. Draw a graph. Use the shell method. The two curves intersect when $0 = x - x^2$. This happens for $x = 0, 1$. Since we are rotating about the line $x = 7$ the radius of each shell will be $7-x$. The height of the shell will be given by $x - x^2$. Therefore the integral is

$$\int_0^1 2\pi(7-x)(x-x^2)\ dx = 2\pi \int_0^1 (7-x)(x-x^2)\ dx.$$

24. The plane region bounded by the curves $y = 2$ and $y = 2+2x-x^2$ is rotated about the x axis. What integral gives the volume?

Solution. Draw a picture. Use the disk method. The two curves intersect when $2 = 2 + 2x - x^2$. Then $x^2 - 2x = 0$, $x(x-2) = 0$, and hence $x = 0, 2$. By testing a value, one sees that $y = 2 + 2x - x^2$ is above $y = 2$ for x between 0 and 2. Since we are rotating about the x-axis, the inner radius is 2 and the outer radius is $2 + 2x - x^2$. Hence the correct integral is

$$\int_0^2 \pi\Big((2+2x-x^2)^2 - 2^2\Big)dx = \pi \int_0^2 \Big((2+2x-x^2)^2 - 4\Big)dx.$$

25. The function $f(x) = \sqrt{16-2x}$ is continuous on the interval $[0, 8]$. What is its average value on this interval?

Solution. By definition the average value is $\frac{1}{8}\int_0^8 \sqrt{16-2x}\ dx$. Toward finding the antiderivative of the integrand, we make the substitution $u = 16 - 2x$. $du = -2$.

$$\int \sqrt{16-2x}\ dx = -\frac{1}{2}\int \sqrt{u}\ du = -\frac{1}{2}\frac{2}{3}u^{3/2} = -\frac{1}{3}(16-2x)^{3/2}.$$

Going back to our expression for average value, we have

$$\frac{1}{8}\int_0^8 \sqrt{16-2x}\ dx = -\frac{1}{8}\frac{1}{3}(16-2x)^{3/2}|_0^8 = -\frac{1}{24}(0-16^{3/2}) = \frac{64}{24} = \frac{8}{3}.$$